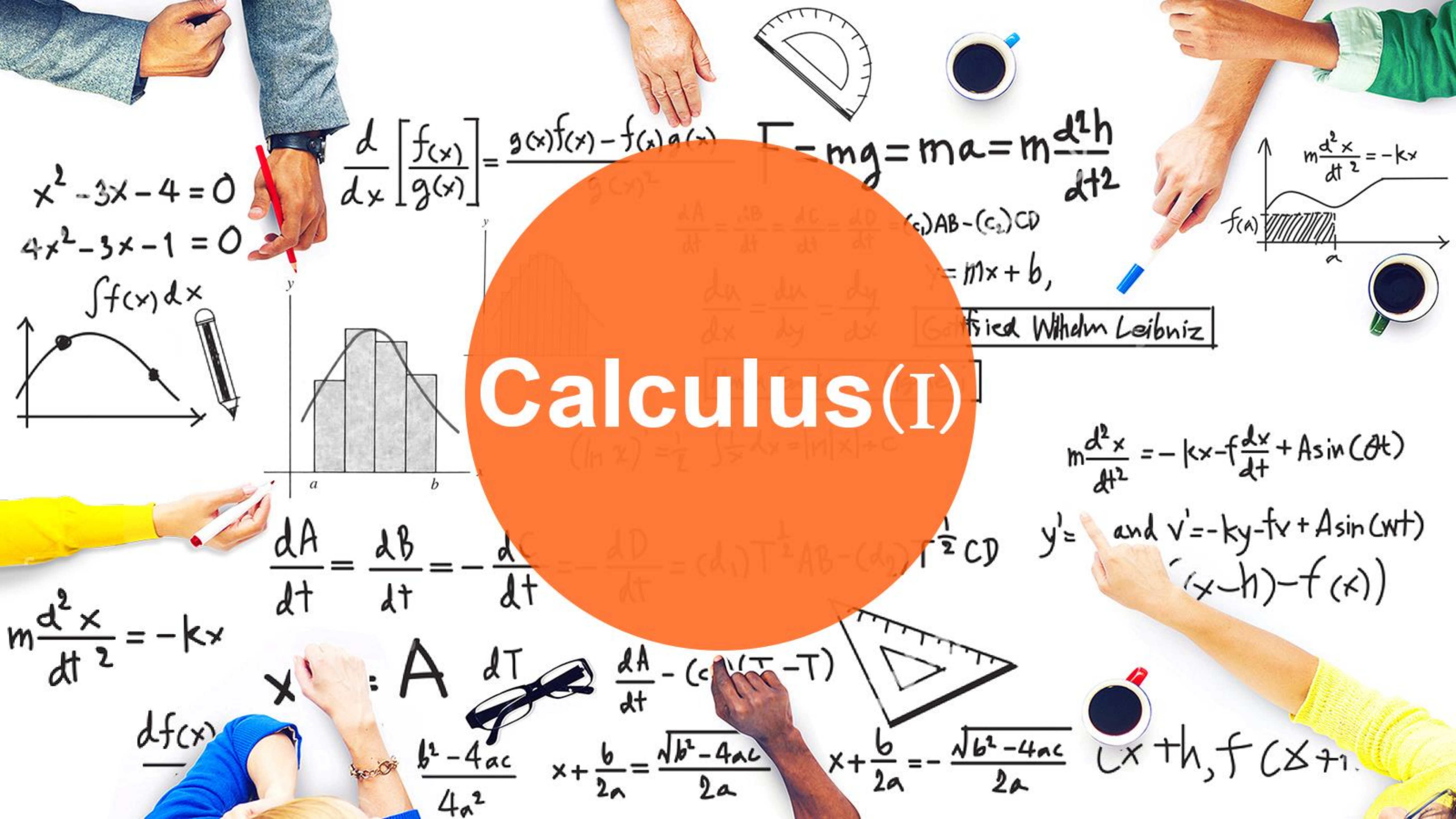


# Calculus(Ⅰ)

# Limits at Infinity; Infinite Limits

Lecturer: Xue Deng

# Preliminaries

D=$N^+$ $\quad y = f(x) \quad (x \in N^+)$

$$u = f(n) \quad (n \in N^+).$$

Natural sequential arrangement for the function values of $u_n$ :

$$\{u_n\} = u_1, u_2, \cdots, u_n, \cdots$$

Sequence:$u_n = f(n)$, general term.

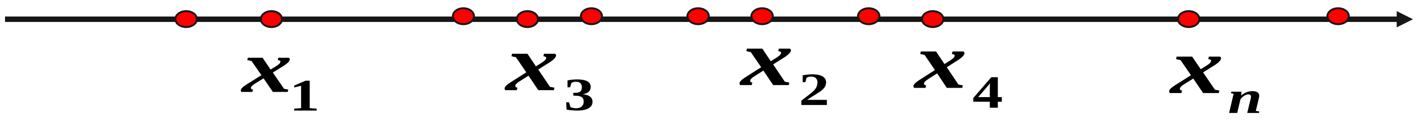
# Preliminaries

## Geometrical meaning

The sequence corresponds to a point sequence of number-axis：

May be regarded as a point set on the axis in order to take

$$x_1, x_2, \cdots, x_n, \cdots.$$



# Preliminaries

Eg

## General Term ?

$$1, \frac{2}{3}, \frac{3}{5}, \cdots;$$ $$\{\frac{n}{2n-1}\}$$

$$-1, 1, -1, 1, \cdots$$ $$\{(-1)^n\}$$

$$2, 4, 8, 16, \cdots$$ $$\{2^n\}$$

$$2, \frac{1}{2}, \frac{4}{3}, \cdots, \frac{n+(-1)^{n-1}}{n}, \cdots;$$ $$\{\frac{n+(-1)^{n-1}}{n}\}$$

# Preliminaries

## a Bounded

As to $u_n, \exists M > 0, s.t. \forall n \in N$, we always have $|u_n| \le M$, then $u_n$ is bounded.

$$u_n = \frac{n}{2n-1}, \quad (-1)^{n-1}, \quad (-\frac{1}{3})^{n-1}, \quad \frac{n+(-1)^{n-1}}{n}$$ **Bounded**

$u_n = 2^n$ **Unbounded.**

$\exists s, G \in R(s < G)$, have $s \le u_n \le G, u_n$ is bounded.

# Preliminaries

b Monotonicity

$$u_n, \quad \forall n, \quad \text{have} \quad u_n \le u_{n+1}$$

$\ge$

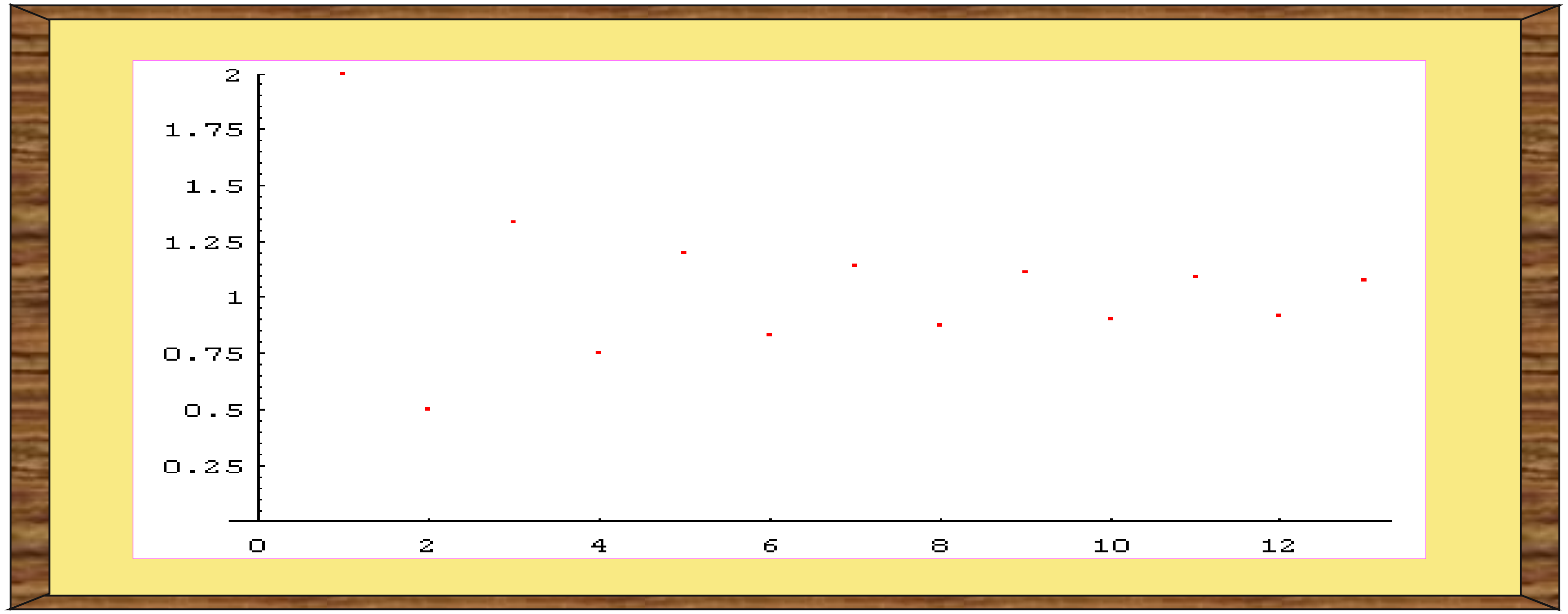
$u_n$ is increasing.

decreasing

# Preliminaries

When $n \to \infty$, $\{1+\frac{(-1)^{n-1}}{n}\}$ how to change?

$$2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5} \cdots$$



# Preliminaries

$$|u_n - 1| = \frac{1}{n}$$

measurement

- **Given** $\frac{1}{100}$, by $\frac{1}{n} < \frac{1}{100}$, need $n > 100$, have $|u_n - 1| < \frac{1}{100}$,

- **Given** $\frac{1}{1000}$, need $n > 1000$, have $|u_n - 1| < \frac{1}{1000}$,

- **Given** $\frac{1}{10000}$, need $n > 10000$, have $|u_n - 1| < \frac{1}{10000}$,

$\forall \varepsilon > 0$, take $n > N(=[\frac{1}{\varepsilon}])$, have $|u_n - 1| < \varepsilon$.

# Preliminaries

Def

$\{u_n\}$ and $A$. $\forall \varepsilon > 0, \exists N > 0,$ s. t. $n > N$, have

$$|u_n - A| < \varepsilon$$

$$\lim_{n\to\infty} u_n = A \text{ OR when } n \to \infty, \ u_n \to A.$$

Converges to $A$, otherwise diverge.

# Preliminaries

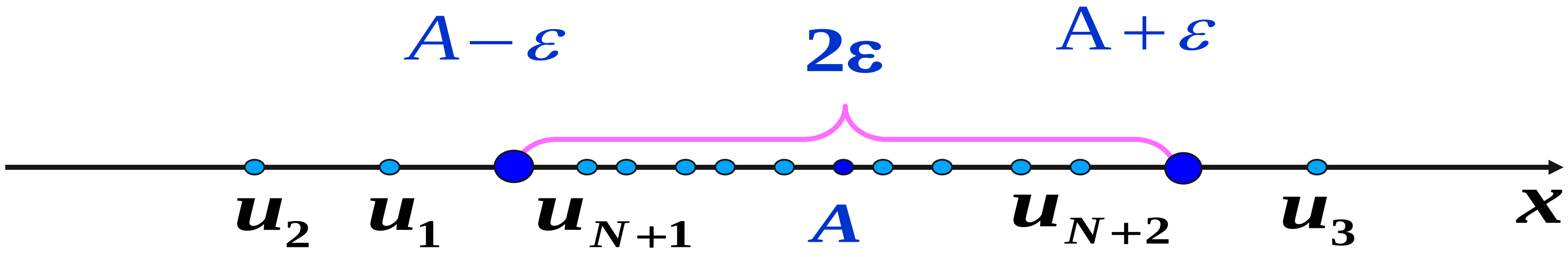
(1) $\forall \varepsilon$: Describe the infinite proximity of $u_n$ and $A$;

(2) $N$ NOT unique : $N(\varepsilon)$.

(3) $\varepsilon - N$: $\lim_{n\to\infty} u_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N > 0,$ $s.t.$ when $n > N$,

$$|u_n - A| < \varepsilon.$$

(4) Geometrical meaning:



# Preliminaries

The definition of a sequence limit does not give the method of finding limit.

**Eg :** Prove $\lim\limits_{n\to\infty}\frac{n+(-1)^{n-1}}{n}=1.$

(1) $\forall \varepsilon > 0,$

$$|u_n - 1| = \left|\frac{n+(-1)^{n-1}}{n} - 1\right| = \frac{1}{n}$$

Give the measurement

(2) **Need** $|u_n - 1| < \varepsilon$, **need** $\frac{1}{n} < \varepsilon$, **or** $n > \frac{1}{\varepsilon}$,

**Find $N$**

(3) So, **take** $N = [\frac{1}{\varepsilon}]$, then $n > N$, we have

$$\left|\frac{n+(-1)^{n-1}}{n} - 1\right| < \varepsilon$$

Definition form

(4) $\therefore$ $\lim\limits_{n\to\infty}\frac{n+(-1)^{n-1}}{n}=1.$

Summary

# Preliminaries

Eg : Prove $\lim_{n\to\infty} q^n = 0$, where $0 < |q| < 1$. Typical limit

$\forall \ \varepsilon > 0$, $|u_n - 0| = |q^n| < \varepsilon$,

need $|u_n - 0| < \varepsilon$, only need $n \ln|q| < \ln \varepsilon$, or $n > \dfrac{\ln \varepsilon}{\ln|q|}$.

Take $N = [\dfrac{\ln \varepsilon}{\ln|q|}]$, when $n > N$, we have

$$|q^n - 0| < \varepsilon,$$

$\therefore$ $\lim_{n\to\infty} q^n = 0.$

# Preliminaries

Simplified proof Even more simplified as

$$\forall \varepsilon > 0, \quad \text{let} \quad N = [\frac{\ln \varepsilon}{\ln |q|}], \quad \text{when} \quad n > N, \text{we have}$$

$$|q^n - 0| < \varepsilon.$$

$$\therefore \qquad \lim_{n\to\infty} q^n = 0.$$

# Limits at Infinity; Infinite Limits